TORSION OF A CIRCULAR BAR WITH TWO CUTS

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This paper deals with the study of a homogeneous isotropic circular bar having two cuts (cracks) along the diameter. The solution is obtained by means of conformal mapping [1].

1. The Mapping Function. The conformal mapping of the circle $|\zeta| < 1$ onto the unit circle with two cuts along the diameter from point +1 to α_1 and from point −1 to α_2 (Fig. 1) is accomplished by the function

$$z = \sqrt{\frac{d}{a}}\,\frac{1 + 2a\zeta + \zeta^2 - b\sqrt{1 + 2c\zeta^2 + \zeta^4}}{1 + 2d\zeta + \zeta^2} \tag{1.1}$$

where

$$a = \cos\beta\cos\delta, \qquad b = \sin\delta, \qquad c = -\cos 2\beta, \qquad d = \frac{\cos\beta}{\cos\delta} \tag{1.2}$$

$$\cos\beta = \frac{2m}{1+m^2}, \quad \sin\beta = \frac{1-m^2}{1+m^2}, \quad \cos\delta = \frac{2n}{1+n^2}, \quad \sin\delta = \frac{1-n^2}{1+n^2}$$

$$m = \frac{1 - \alpha_1\alpha_2 - \sqrt{(1-\alpha_1^2)(1-\alpha_2^2)}}{\alpha_1 - \alpha_2}, \qquad n = \frac{1 + \alpha_1\alpha_2 - \sqrt{(1-\alpha_1^2)(1-\alpha_2^2)}}{\alpha_1 + \alpha_2} \tag{1.3}$$

Suppose that $0 < \alpha_1 < 1$; $\alpha_1 > |\alpha_2|$, i.e. it will be assumed that the depth of the "left" cut is greater than the depth of the "right" cut. Then $0 < m < 1$, $0 < n < 1$, (Fig. 1).

The case $n = 0$ is that of two cuts of equal length.

The roots of the denominator in expression (1.1) will be denoted by ζ_1 and ζ_2. If $n > m$, then ζ_1 and ζ_2 are complex conjugates, $|\zeta_1| = |\zeta_2| = 1$. For $n = m$ one has $\zeta_1 = \zeta_2 = -1$. If $n < m$, then the roots ζ_1 and ζ_2 are real and different, $|\zeta_1| < 1$, $|\zeta_2| > 1$.

For $\zeta = \zeta_1$, the numerator of expression (1.1) has to reduce to zero, which is the condition for the choice of the branch of the radical, appearing in (1.1). The zeros of the expression under the radical in (1.1) will be denoted by z_1, z_2, z_3, z_4.

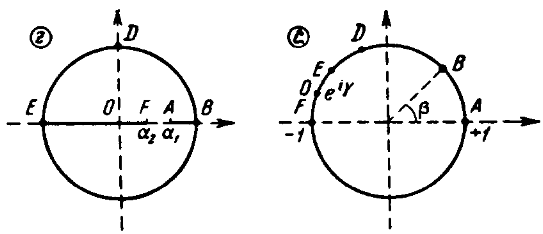

Fig. 1.

2. The Complex Torsion Function. 1. Let $n > m$, then $a_2 > 0$. Introduce the parameter γ:

$$\cos\gamma = -\frac{(1+n^2)m}{(1+m^2)n}, \qquad \sin\gamma = \frac{\sqrt{(n^2-m^2)(1-n^2m^2)}}{(1+m^2)n} \tag{2.1}$$

yielding

$$d = -\cos\gamma, \quad \zeta_1 = e^{i\gamma}, \ z_1 = e^{i\beta}, \ z_2 = \bar{z}_1, \ z_3 = -\bar{z}_1, \ z_4 = -z_1 \tag{2.2}$$

Proceeding similarly as in an earlier paper [3], the following expression can be found for the complex torsion function:

$$f(\zeta) = \frac{b^2 di}{\pi a(1+2d\zeta+\zeta^2)^2}\Big[P + Q\zeta + R\zeta^2 + S\zeta^3 - \frac{\pi}{b}(1+2a\zeta+\zeta^2)\sqrt{1+2c\zeta^2+\zeta^4} + $$
$$+ i(1+2c\zeta^2+\zeta^4)\log\frac{1-z_1^2\zeta^2}{1-z_2^2\zeta^2}\Big] + \text{const} \tag{2.3}$$

where

$$P = 2\left[\operatorname{ctg}\gamma\left(1+\frac{\sin^2\beta}{\sin^2\gamma}\right)\log\frac{\sin(\beta+\gamma)}{\sin(\beta-\gamma)} - \frac{\sin 2\beta}{\sin^2\gamma}\right] \tag{2.4}$$

$$Q = 2\left[\left(2\sin\gamma - \frac{3}{\sin\gamma} - \frac{3\sin^2\beta}{\sin^3\gamma} + \frac{4\sin^2\beta}{\sin\gamma}\right)\log\frac{\sin(\beta+\gamma)}{\sin(\beta-\gamma)} + 4\beta\cos\gamma + 3\cos\gamma\frac{\sin 2\beta}{\sin^2\gamma}\right]$$

$$R = (1+2\cos^2\gamma)P - 8\beta(\cos^2\beta+\cos^2\gamma), \qquad S = 16\beta\cos\gamma - Q - 4P\cos\gamma$$

and it is to be understood that the branch of the logarithm is represented by the series

$$\log\frac{1-z_1^2\zeta^2}{1-z_2^2\zeta^2} = -2i\left(\sin 2\beta\,\zeta^2 + \frac{\sin 4\beta}{2}\zeta^4 + \frac{\sin 6\beta}{3}\zeta^6 + \dots\right) \tag{2.5}$$

2. Let $n = m$, then $a_2 = 0$, $\gamma = \pi$, $\delta = \beta$. The expression for the complex torsion function is found by substituting $d = 1$ in (2.3) and replacing P, Q, R, S by their limiting values for $\gamma \to \pi$. Note that

$$\lim_{\gamma\to\pi} P = \frac{16}{3}\operatorname{ctg}\beta - \frac{2}{3}\sin 2\beta, \qquad \lim_{\gamma\to\pi} R = 16\operatorname{ctg}\beta - 2\sin 2\beta - 8\beta(1+\cos^2\beta)$$

$$\lim_{\gamma\to\pi} Q = 16\operatorname{ctg}\beta - 4\sin 2\beta - 8\beta, \qquad \lim_{\gamma\to\pi} S = \frac{16}{3}\operatorname{ctg}\beta + \frac{4}{3}\sin 2\beta - 8\beta \tag{2.6}$$

3. Now let $n < m$, then $a_2 < 0$. In (1.1) carry out an additional transformation of the plane ζ :

$$\zeta^* = \frac{\zeta - \zeta_1}{1 - \zeta_1 \zeta} \tag{2.7}$$

As a result (again subsituting ζ for ζ^*) we find:

$$z = \frac{b_1}{\zeta}\left(1 - 2a_1\zeta + \zeta^2 - \sqrt{1 - 4a_1\zeta + 2c_1\zeta^2 - 4a_1\zeta^3 + \zeta^4}\right) \tag{2.8}$$

where

$$a_1 = \frac{1}{2}(\cos\beta_1 + \cos\beta_2), \qquad b_1 = \frac{1}{\cos\beta_1 - \cos\beta_2}, \qquad c_1 = 1 + 2\cos\beta_1\cos\beta_2 \tag{2.9}$$

$$\cos\beta_1 = \frac{n(1+m^2)^2 + 2m^2(1+n^2)}{m(1+m^2)(1+n)^2}; \qquad \sin\beta_1 = \frac{(1-m^2)\sqrt{(m^2-n^2)(1-m^2n^2)}}{m(1+m^2)(1+n)^2}$$

$$\cos\beta_2 = \frac{n(1+m^2)^2 - 2m^2(1+n^2)}{m(1+m^2)(1-n)^2}; \qquad \sin\beta_2 = \frac{(1-m^2)\sqrt{(m^2-n^2)(1-m^2n^2)}}{m(1+m^2)(1-n)^2} \tag{2.10}$$

The zeros of the expression under the radical in (2.8) will be

$$z_1 = e^{i\beta_1}, \qquad z_2 = \bar{z}_1, \qquad z_3 = e^{i\beta_2}, \qquad z_4 = \bar{z}_3$$

The following expression is obtained for the complex torsion function:

$$f(\zeta) = ib_1^2\left[\frac{1}{\zeta^2} + \frac{p}{\zeta} + q\zeta + r\zeta^2 - \left(\frac{1}{\zeta^2} - \frac{2a_1}{\zeta} + 1\right)\sqrt{1 - 4a_1\zeta + 2c_1\zeta^2 - 4a_1\zeta^3 + \zeta^4} + \right.$$
$$\left. + \frac{i}{\pi\zeta^2}\left(1 - 4a_1\zeta + 2c_1\zeta^2 - 4a_1\zeta^3 + \zeta^4\right)\log\frac{(1 - z_1\zeta)(1 - z_4\zeta)}{(1 - z_2\zeta)(1 - z_3\zeta)}\right] + \text{const} \tag{2.11}$$

where

$$p = \frac{2}{\pi}(\sin\beta_2 - \sin\beta_1) - 4a_1 \qquad q = \frac{8}{\pi}a_1(\beta_2 - \beta_1) - \frac{2}{\pi}(\sin\beta_2 - \sin\beta_1) - 4a_1 \tag{2.12}$$

$$r = 1 - \frac{2}{\pi}(\beta_2 - \beta_1)$$

4. Two equal cuts. Let $n = 0$, then $a_1 = -a_2 = m$. Noting that $\beta_1 = \beta$, $\beta_2 = \pi - \beta$, it is found from (2.11) that

$$f(\zeta) = \frac{i}{4\cos^2\beta}\left[\frac{1}{\zeta^2} + \left(1 - \frac{8}{\pi}\operatorname{arc\,tg} m\right)\zeta^2 - \left(1 + \frac{1}{\zeta^2}\right)\sqrt{1 + 2c\zeta^2 + \zeta^4} + \right.$$
$$\left. + i\,\frac{1 + 2c\zeta^2 + \zeta^4}{\pi\zeta^2}\log\frac{1 - z_1^2\zeta^2}{1 - z_2^2\zeta^2}\right] + \text{const} \tag{2.13}$$

3. Stresses on the Circular Part of the Contour. The following expression is obtained for the tangential stresses on the circular part of the contour of a transverse cross-section of the bar in the case $n > m$:

$$T_\vartheta = \frac{2\mu\tau\sqrt{\sin^2\vartheta - \sin^2\beta}}{(\cos\vartheta - \cos\gamma)^2}\left[\operatorname{tg}^2\delta\sqrt{\sin^2\vartheta - \sin^2\beta} - \frac{\sin\beta\operatorname{tg}\delta}{\pi}\frac{\cos\vartheta - \cos\gamma}{\sin\vartheta} + \right.$$
$$\left. + \frac{\sin\delta}{\pi\cos^2\delta}(\cos\vartheta + \cos\beta\cos\delta)\log\frac{\sin(\vartheta+\beta)}{\sin(\vartheta-\beta)} - \frac{\operatorname{tg}\delta}{2\pi\cos\beta}\frac{M\cos 2\vartheta + N\cos\vartheta + K}{\sin\vartheta}\right]$$
$$\pi - \beta > \vartheta > \beta \tag{3.1}$$

where τ is the amount of twist, μ is Lame's constant,

$$
\begin{aligned}
M &= \frac{1}{2}\left[\left(\frac{\sin^2\beta}{\sin^3\gamma} - 2\sin\gamma + \frac{1}{\sin\gamma}\right)\log\frac{\sin(\beta+\gamma)}{\sin(\beta-\gamma)} - \frac{\sin 2\beta}{\sin\gamma}\operatorname{ctg}\gamma\right] \\
N &= \left[\sin 2\beta + 2\sin 2\beta\operatorname{ctg}^2\gamma - 2(\operatorname{ctg}\gamma + \sin^2\beta\operatorname{ctg}^3\gamma)\log\frac{\sin(\beta+\gamma)}{\sin(\beta-\gamma)}\right] \\
K &= \frac{3}{2}\left[\left(\frac{\sin^2\beta}{\sin^3\gamma} + \frac{1}{\sin\gamma} - \frac{2\sin^2\beta}{\sin\gamma}\right)\log\frac{\sin(\beta+\gamma)}{\sin(\beta-\gamma)} - \cos\gamma\frac{\sin 2\beta}{\sin^2\gamma}\right]
\end{aligned}
\tag{3.2}
$$

The stress will be found at point D (Fig. 1) where $z = i$. The point $\sigma_1 = \cos\theta_1 + i\sin\theta_1$ is the corresponding point in the ζ plane, where $\cos\theta_1 = -\cos\beta\cos\delta$, $\sin\theta_1 = \sqrt{1-\cos^2\beta\cos^2\delta}$. Letting $\theta = \theta_1$ in (3.1) yields

$$
\frac{1}{\mu\tau}T_{\vartheta_1} = 2 - \frac{1}{\pi\sqrt{1-\cos^2\beta\cos^2\delta}}\left[2\sin\beta\left(1 + \frac{1}{\sin^2\gamma} + \frac{\cos^2\delta\cos^2\beta}{\sin^2\delta\sin^2\gamma}\right) + \right.
$$
$$
\left. + \frac{\cos^2\gamma\sin^4\delta}{\sin^3\gamma\cos\delta}\log\frac{\sin(\beta+\gamma)}{\sin(\beta-\gamma)}\right]
\tag{3.3}
$$

The case $m = 0$ ($\beta = 1/2\pi$, $\gamma = 1/2\pi$) is the limiting case, when the cuts meet at the point $z = n$, and two bars of semicircular cross-section are obtained. Letting $m = 0$ in (3.3) yields

$$
T_{\vartheta_1} = \left(2 - \frac{4}{\pi}\right)\mu\tau
$$

which, in fact, is the expression that should hold for a semicircular cross-section.

The stress on the circular part of the contour for the case $n = m$ can be easily found by substituting $\gamma = \pi$ in (3.1) and replacing M, N, K by their limiting values for $\gamma \to \pi$.

Starting from (2.11), one can obtain the expression for the tangential stresses on the circular part of the contour for the case $n < m$.

In the case of two equal cuts ($n = 0$), starting from (2.13), one obtains

$$
T_\vartheta = \frac{\mu\tau}{\cos^2\beta}\sqrt{\sin^2\vartheta - \sin^2\beta}\left[\left(1 - \frac{2\beta}{\pi}\right)\frac{\cos 2\vartheta}{\sin\vartheta} - \frac{\sin 2\beta}{\pi\sin\vartheta} + \frac{2}{\pi}\cos\vartheta\log\frac{\sin(\vartheta+\beta)}{\sin(\vartheta-\beta)} + \right.
$$
$$
\left. + 2\sqrt{\sin^2\vartheta - \sin^2\beta}\right], \qquad \pi - \beta > \vartheta > \beta
\tag{3.4}
$$

The maximum stress in the case of two equal cuts is reached at point D with $\theta = 1/2\pi$.

$$
\frac{1}{\mu\tau}T_{\max} = 2 - \frac{2}{\pi}\frac{1-m^2}{1+m^2} - \frac{2}{\pi}\frac{1+m^2}{m}\operatorname{arc\,tg} m
\tag{3.5}
$$

4. Stress at the Edges of the Cuts. In the case $n > m$ the following tangential stresses $T_\theta^{(1)}$ at the edges of the cuts are obtained:

$$T_\vartheta^{(1)} = \frac{\mu\tau\sin\delta}{\pi\cos\beta}\,\frac{\sqrt{\sin^2\beta-\sin^2\vartheta}}{\cos\beta\cos\delta+\cos\vartheta-\sin\delta\sqrt{\sin^2\beta-\sin^2\vartheta}}\times$$

$$\times\left[\frac{2(\cos^2\beta-\cos\gamma\cos\vartheta)}{\cos\vartheta-\cos\gamma}\log\frac{\sin(\beta+\vartheta)}{\sin(\beta-\vartheta)}-\frac{\sin 2\beta}{\sin\vartheta}-\frac{M\cos 2\vartheta+N\cos\vartheta+K}{\sin\vartheta(\cos\vartheta-\cos\gamma)}\right] \qquad (4.1)$$

$$\beta>\vartheta>0 \quad \text{or} \quad \pi>\vartheta>\pi-\beta$$

The stress at point O (Fig. 1), $z = 0$, is to be determined. In the ζ plane, this corresponds to the point $\zeta = e^{i\gamma}$. Substituting $\theta = \gamma$ into (4.1) yields (evaluating the indeterminate expression):

$$T_\gamma^{(1)} = \frac{2\mu\tau}{3\pi}\sin^2\beta\left[\frac{4\sin\beta}{\sin^3\gamma}+\frac{3\cos\gamma\sin^2\delta\sin 2\beta}{\cos^3\delta\sin^5\gamma}-\frac{3\sin^4\delta\cos^2\gamma}{\cos^3\delta\sin^6\gamma}\log\frac{\sin(\beta+\gamma)}{\sin(\beta-\gamma)}\right] \qquad (4.2)$$

Letting ■ = 0 (the cuts meet) in the last formula results in

$$T_\gamma^{(1)} = \frac{8}{3\pi}\mu\tau$$

which, in fact, is the proper result for a semicircular section. The stresses at the sides of the cuts for the case n = ■ are easily obtained from (4.1) by letting $\gamma = \pi$.

In the case of two equal cuts starting from (2.13) one obtains

$$T_\vartheta^{(1)} = \frac{\mu\tau}{\pi\cos\beta}\,\frac{\sqrt{\sin^2\beta-\sin^2\vartheta}}{\cos\vartheta-\sqrt{\sin^2\beta-\sin^2\vartheta}}\left[(\pi-2\beta)\frac{\cos 2\vartheta}{\sin\vartheta}-\frac{\sin 2\beta}{\sin\vartheta}+2\cos\vartheta\log\frac{\sin(\beta+\vartheta)}{\sin(\beta-\vartheta)}\right]$$

$$\beta>\vartheta>0 \quad \text{or} \quad \pi>\vartheta>\pi-\beta \qquad (4.3)$$

5. Torsional Rigidity. The following expression is obtained for the torsional rigidity in the case of two equal cuts:

$$\frac{1}{\mu}C = \pi-\frac{1}{2\pi}\,\frac{1}{\cos^4\beta}\left[(\pi-2\beta)^2+(\pi-2\beta)\sin 4\beta+\sin^2 2\beta\right] \qquad (5.1)$$

Substituting in (5.1) $\beta = 1/4\,\pi$ (■ = $\sqrt{2} - 1$) one can find $C = 1/2\,\pi - 2/\pi$. The limiting case ■ = 0 ($\beta = 1/2\,\pi$) yields $C = (\pi - 8/\pi)\mu$, equal to twice the value of the torsional rigidity of a beam of semicircular section, which, in fact, it should be. For ■ = 1 ($\beta = 0$) one gets $C = 1/2\,\pi\mu$, i.e. the rigidity of a circular bar without cuts.

The problem of torsion of a circular bar with two equal cuts has been solved by another method in a paper by Shepherd [2]. The torsional rigidity was calculated in two cases: for 1 - ■ = 0.1591 and 1 - ■ = 0.2929.

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